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QUANTUM RELATIVISTIC EQUATION FOR A PROBABILTY AMPLITUDE

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Abstract

The relativistic quantum equation is proposed for the complex wave function, which has the meaning of a probability amplitude. The Lagrangian formulation of the proposed theory is developed. The problem of spreading of a wave packet in an unlimited space is solved. The relativistic correction to the energy levels of a harmonic oscillator is found, leading to a violation of their equidistance.

Key words: Schrödinger equation, quantum mechanics, relativistic equation, wave function, probability amplitude, harmonic oscillator

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1 Introduction

Fundamental issues related to the completeness of the description of reality in quantum mechanics were discussed long ago in well-known works [1,2] and then in many others. These discussions continue in present time. The Schrödinger equation for the complex wave function ψ, which, according to Born's interpretation, has the meaning of a probability amplitude, is nonrelativistic. This means that the theory allows propagation of signals at arbitrarily high speeds. Perhaps, some of the discussed difficulties and paradoxes of quantum mechanics are related to this circumstance. It should also be noted that quantum theory is used in the description and interpretation of experiments with photons, which are relativistic objects. So far there is no generally accepted relativistic equation for the field, which could be interpreted as a probability amplitude and would allow a Born probabilistic interpretation. The probabilistic interpretation of the complex field within the framework of Dirac theory was discussed in works [3,4].

In this paper the relativistic equation is proposed for the complex scalar field, allowing its physical interpretation as a probability amplitude. The theory is formulated within the framework of the Lagrangian formalism. The spreading of a wave packet in an unlimited space is considered. The relativistic correction has been found in the theory of a harmonic oscillator, leading to a violation of the equidistance of levels. The issue of the completeness of the quantum description is not addressed in the paper.

2 Relativistic equation for the complex scalar field

The recipe for the transition from the classical to the quantum mechanical equation consists, as is known [5], in replacing momentum and energy in the classical Hamiltonian $H = \mathbf{p}^2/2m$ with the operators $\mathbf{p} \to -i\hbar\nabla$ and $H \to i\hbar\,\partial/\partial t$, which act on the complex function ψ. In the relativistic case, the relationship between energy and momentum is given by formula $H = \sqrt{m^2c^4 + c^2\mathbf{p}^2}$, where m is the particle mass, c is the speed of light. Using the indicated substitution, we arrive at the relativistic equation

$$i\hbar\frac{\partial\psi}{\partial t} = \sigma\sqrt{m^2c^4 - \hbar^2c^2\Delta}\;\psi, \qquad (1)$$

where Δ is the Laplace operator, $\sigma = \pm 1$. This equation is inconvenient for a number of reasons. Firstly, the time and space coordinates enter here unequally, so that it does not have an explicitly covariant form. Secondly, because in Eq. (1) the Laplace operator appears under the sign of the root, and, therefore, this equation is nonlocal. In order to get rid of the noted formal shortcomings, the transition to the quantum mechanical equation is usually carried out in the expression for the square of energy $H^2 = m^2c^4 + c^2\mathbf{p}^2$, which leads to the Klein-Gordon-Fock equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \lambda\!\!\!^{-}{}^{-2}\right)\psi = 0, \tag{2}$$

where $\lambda\!\!\!^{-} \equiv \hbar/mc$ is the Compton length. In the following we will also use 4-dimensional notation, in which an arbitrary 4-vector is $B \equiv B_\mu \equiv (\mathbf{B}, B_4 = iB_0)$, in particular $x \equiv x_\mu \equiv (\mathbf{x}, x_4 = ix_0 = ict)$. The scalar product of two vectors is written as $AB \equiv A_\mu B_\mu = \mathbf{AB} + A_4B_4 = \mathbf{AB} - A_0B_0$. A summation from 1 to 4 is implied over repeating Greek indices, and a summation from 1 to 3 – over repeating Latin indices. In this notation equation (2) can be written in the explicitly covariant form

$$\left(\frac{\partial^2}{\partial x_\mu^2} - \lambda\!\!\!^{-}{}^{-2}\right)\psi(x) = 0. \tag{3}$$

From this equation there follows the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{4}$$

where

$$\rho = i\frac{\lambda\!\!\!^{-}}{2c}\left(\psi^*\frac{\partial \psi}{\partial t} - \psi\frac{\partial \psi^*}{\partial t}\right), \qquad \mathbf{j} = -i\frac{\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*). \tag{5}$$

Here the quantity ρ is not positively definite, and therefore cannot be interpreted as a probability density. In addition, the transition to a higher order equation leads to the emergence of new solutions. In connection with this, the problem arises of constructing the quantum relativistic equation for the complex field, which would have the meaning of a probability amplitude.

3 Relativistic quantum equation for a probability amplitude

In spite of the shortcomings noted above, let us consider equation (1) as the equation for the probability amplitude $\psi(x)$. The advantage of this equation, which to a great extent compensates for the difficulties noted above, consists in the possibility of interpreting the complex field as a probability amplitude. In this equation the sign in front of the root can be chosen arbitrarily, since this will not affect the physical results, but only affect the form of the time dependence of the wave function for stationary states $\psi \sim \exp\bigl(-i\sigma Et/\hbar\bigr)$. In order for equation (1) to have the generally accepted form of the Schrödinger equation in the nonrelativistic limit, we will assume in (1) $\sigma = +1$. The equation

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{m^2c^4 - \hbar^2c^2\Delta}\;\psi \tag{6}$$

does not have an explicitly covariant form, but leads to the continuity equation for the probability density $|\psi|^2$:

$$\frac{\partial|\psi|^2}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{7}$$

where

$$\nabla\cdot\mathbf{j} = i\frac{mc^2}{\hbar}\Bigl(\psi^*\sqrt{1-\lambda^2\Delta}\;\psi - \psi\sqrt{1-\lambda^2\Delta}\;\psi^*\Bigr), \tag{8}$$

and $\mathbf{j}$ has the meaning of the probability flux density. The continuity equation (7) for the probability density can be written in 4-dimensional form

$$\frac{\partial j_\mu(x)}{\partial x_\mu} = 0, \tag{9}$$

where $j_\mu \equiv \bigl(\mathbf{j}, ic|\psi|^2\bigr)$ is a 4-vector of flux density, so that it has the same form in any inertial reference system. Thus, the equation for the probability amplitude (6), written in an arbitrary reference system, leads to the Lorentz covariant form of the probability conservation law (9). Since under the Lorentz transformation $x_\mu \to x'_\mu = a_{\mu\nu}x_\nu$, where $a_{\mu\nu}a_{\mu\nu'} = \delta_{\nu\nu'}$, the 4-vector is transformed according to the law $j'_\mu(x') = a_{\mu\nu}j_\nu(x)$, then the probability

density and the probability flux density in different systems are connected by the relations

$$\begin{aligned} ic|\psi'(x')|^2 &= ic|\psi(x)|^2 + a_{4m}j_m(x), \\ j'_k(x') &= a_{k4}\, ic|\psi(x)|^2 + a_{km}j_m(x). \end{aligned} \tag{10}$$

Let us represent the complex field as the sum of the real and imaginary parts

$$\psi(x) = \frac{1}{\sqrt{2}}\big[\psi'(x) + i\psi''(x)\big]. \tag{11}$$

Then equation (6) turns out to be equivalent to a system of two equations for the real fields ψ' and ψ'':

$$\hbar\dot{\psi}' = \sqrt{m^2c^4 - \hbar^2c^2\Delta}\ \psi'', \qquad \hbar\dot{\psi}'' = -\sqrt{m^2c^4 - \hbar^2c^2\Delta}\ \psi'. \tag{12}$$

Here and henceforth we also use the notation $\dot{\psi} \equiv \partial\psi/\partial t$. One of the functions, for example ψ'', can be eliminated, and then we arrive at the explicitly covariant Klein-Gordon-Fock equation for the real part

$$\ddot{\psi}' - c^2\Delta\psi' + c^2\lambda\!\!\!^{-2}\psi' = 0. \tag{13}$$

Along with this, the evolution of the imaginary part is determined by the second equation (12). Similarly, the Klein-Gordon-Fock equation can be obtained for the imaginary part of the function (11).

Let us consider the meaning of the expression $\sqrt{1 - \lambda\!\!\!^{2}\Delta}\ \psi$, containing the square root of the Laplace operator. For $x^2 < 1$ the following expansions are valid [6]

$$\begin{aligned} \sqrt{1 \pm x} &= 1 - \sum_{n=1}^{\infty}(\mp)^n a_n x^n, \\ a_n \equiv \frac{(2n-3)!!}{(2n)!!}, &\quad a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{8}, \quad a_3 = \frac{3}{48}, \end{aligned} \tag{14}$$

$$\begin{aligned} (1 \pm x)^{-1/2} &= 1 + \sum_{n=1}^{\infty}(\mp)^n b_n x^n, \\ b_n \equiv \frac{(2n-1)!!}{(2n)!!}, &\quad b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{8}, \quad b_3 = \frac{15}{48}, \end{aligned} \tag{15}$$

where $(2n)!! = 2 \cdot 4 \cdot 6 \ldots 2n = 2^n n!$, $(2n+1)!! = 1 \cdot 3 \cdot 5 \ldots (2n+1) = \frac{2^{n+1}}{\sqrt{\pi}} \Gamma\left(n+3/2\right)$. With allowance for these representations, by the expression with the square root from the operator we will understand the following expansion

$$\sqrt{1-\lambda\!\!\!^{-2}\Delta}\ \psi = \left(1 - \sum_{n=1}^{\infty} a_n \lambda\!\!\!^{-2n} \Delta^n\right)\psi. \tag{16}$$

This expansion is valid under the condition $|\lambda\!\!\!^{-2}\Delta\psi| < 1$, which is assumed to be satisfied in the following. This means that one considers the states in which the wave function changes over distances greater than the Compton length. The opposite ultrarelativistic case can be considered in a similar way. It should be noted, however, that it is of rather methodological interest, since in this limit the possibility of transformation of particles should be taken into account, which can only be done by passing to the representation of secondary quantization.

Taking (16) into account, we obtain a formula for the probability flux density. The divergence (8) can be written in the form

$$\nabla \cdot \mathbf{j} = -i\frac{mc^2}{\hbar} \sum_{n=1}^{\infty} a_n \lambda\!\!\!^{-2n} \left(\psi^* \Delta^n \psi - \psi \Delta^n \psi^*\right). \tag{17}$$

From here it follows that the probability flux density has the form

$$\mathbf{j} = \sum_{n=1}^{\infty} \mathbf{j}^{(n)},$$

$$\mathbf{j}^{(n)} = -ica_n \lambda\!\!\!^{-2n-1} \sum_{\alpha=1}^{n} \left[\left(\Delta^{\alpha-1}\psi^*\right)\left(\nabla\Delta^{n-\alpha}\psi\right) - \left(\Delta^{\alpha-1}\psi\right)\left(\nabla\Delta^{n-\alpha}\psi^*\right)\right]. \tag{18}$$

Everywhere it is assumed that the Laplacian to the zeroth power is equal to unity $\Delta^0 \equiv 1$.

4 Lagrangian formalism

Equation (6) can be obtained within the Lagrange formalism, if we choose the Lagrangian in the form

$$L = \frac{\psi^*}{2}\Big(i\hbar\dot{\psi} - mc^2\sqrt{1-\lambda\!\!\!^{-2}\Delta}\;\psi\Big) - \frac{\psi}{2}\Big(i\hbar\dot{\psi}^* + mc^2\sqrt{1-\lambda\!\!\!^{-2}\Delta}\;\psi^*\Big). \tag{19}$$

Lagrangian (19) is a function of variables $\psi, \psi^*, \dot{\psi}, \dot{\psi}^*$ and derivatives with respect to spatial coordinates $\Delta\psi, \Delta^2\psi, \ldots \Delta^n\psi, \ldots \Delta\psi^*, \Delta^2\psi^*, \ldots \Delta^n\psi^*, \ldots$. In this case, the Euler-Lagrange equations have the form

$$\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\psi}} + \Delta\frac{\partial L}{\partial \Delta\psi} + \ldots + \Delta^n\frac{\partial L}{\partial \Delta^n\psi} + \ldots = 0, \tag{20}$$

and a similar equation with the replacement $\psi \to \psi^*$. A substitution of the Lagrangian (19) into (20) leads to Eq. (6).

In the absence of time-dependent external fields, from the requirement of invariance of the Lagrangian under the time shift $t \to t + t_0$ and $\psi(t) \to \psi'(t + t_0)$, there follows the continuity equation for the energy density

$$\frac{\partial \mathrm{H}}{\partial t} + \nabla \cdot \mathbf{j}_E = 0, \tag{21}$$

where the energy density

$$\mathrm{H} \equiv \dot{\psi}\frac{\partial L}{\partial \dot{\psi}} + \dot{\psi}^*\frac{\partial L}{\partial \dot{\psi}^*} - L = \frac{mc^2}{2}\Big[\psi^*\sqrt{1-\lambda\!\!\!^{-2}\Delta}\;\psi + \psi\sqrt{1-\lambda\!\!\!^{-2}\Delta}\;\psi^*\Big]. \tag{22}$$

The energy flux density is given by the formulas

$$\mathbf{j}_E = \sum_{n=1}^{\infty}\mathbf{j}_E^{(n)},$$

$$\mathbf{j}_E^{(n)} = \sum_{\alpha=1}^{n}\Big[\big(\Delta^{n-\alpha}\dot{\psi}\big)\big(\nabla\Delta^{\alpha-1}L_n\big) - \big(\nabla\Delta^{n-\alpha}\dot{\psi}\big)\big(\Delta^{\alpha-1}L_n\big) + \tag{23}$$

$$+\big(\Delta^{n-\alpha}\dot{\psi}^*\big)\big(\nabla\Delta^{\alpha-1}L_n^*\big) - \big(\nabla\Delta^{n-\alpha}\dot{\psi}^*\big)\big(\Delta^{\alpha-1}L_n^*\big)\Big].$$

Here we used the notation

$$L_n \equiv \frac{\partial L}{\partial \Delta^n \psi} = mc^2 a_n \lambda^{2n} \frac{\psi^*}{2}, \quad L_n^* \equiv \frac{\partial L}{\partial \Delta^n \psi^*} = mc^2 a_n \lambda^{2n} \frac{\psi}{2},$$
$$\Delta^\alpha L_n = mc^2 a_n \lambda^{2n} \frac{\Delta^\alpha \psi^*}{2}, \quad \Delta^\alpha L_n^* = mc^2 a_n \lambda^{2n} \frac{\Delta^\alpha \psi}{2}. \tag{24}$$

Taking (24) into account, we have the final expression for the energy flux density

$$\mathbf{j}_E^{(n)} = \frac{1}{2} mc^2 a_n \lambda^{2n} \sum_{\alpha=1}^{n} \Big[\big(\Delta^{n-\alpha} \dot{\psi}\big) \big(\nabla \Delta^{\alpha-1} \psi^*\big) - \big(\nabla \Delta^{n-\alpha} \dot{\psi}\big) \big(\Delta^{\alpha-1} \psi^*\big) + \\ + \big(\Delta^{n-\alpha} \dot{\psi}^*\big) \big(\nabla \Delta^{\alpha-1} \psi\big) - \big(\nabla \Delta^{n-\alpha} \dot{\psi}^*\big) \big(\Delta^{\alpha-1} \psi\big) \Big]. \tag{25}$$

The condition of invariance of the Lagrangian with respect to spatial translations $\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \mathbf{x}_0$ leads to the continuity equation for the momentum density:

$$\pi_i + \nabla_k \sigma_{ik} = 0, \tag{26}$$

where the momentum density π_i and the flux density of i-component of momentum σ_{ik} are given by the formulas

$$\pi_i = -\nabla_i \psi^* \frac{\partial L}{\partial \dot{\psi}^*} - \nabla_i \psi \frac{\partial L}{\partial \dot{\psi}} = -\frac{i\hbar}{2} \big(\psi^* \nabla_i \psi - \psi \nabla_i \psi^*\big), \tag{27}$$

$$\sigma_{ik} = \sum_{n=1}^{\infty} \sigma_{ik}^{(n)},$$

$$\sigma_{ik}^{(n)} = -\sum_{\alpha=1}^{n} \Big[\big(\Delta^{n-\alpha} \nabla_i \psi\big) \big(\nabla_k \Delta^{\alpha-1} L_n\big) - \big(\nabla_k \Delta^{n-\alpha} \nabla_i \psi\big) \big(\Delta^{\alpha-1} L_n\big) + \\ + \big(\Delta^{n-\alpha} \nabla_i \psi^*\big) \big(\nabla_k \Delta^{\alpha-1} L_n^*\big) - \big(\nabla_k \Delta^{n-\alpha} \nabla_i \psi^*\big) \big(\Delta^{\alpha-1} L_n^*\big) \Big]. \tag{28}$$

With allowance for (24), the last formula takes the form

$$\sigma_{ik}^{(n)} = -\frac{1}{2} mc^2 a_n \lambda^{2n} \times \\ \times \sum_{\alpha=1}^{n} \Big[\big(\Delta^{n-\alpha} \nabla_i \psi\big) \big(\nabla_k \Delta^{\alpha-1} \psi^*\big) - \big(\nabla_k \Delta^{n-\alpha} \nabla_i \psi\big) \big(\Delta^{\alpha-1} \psi^*\big) + \\ + \big(\Delta^{n-\alpha} \nabla_i \psi^*\big) \big(\nabla_k \Delta^{\alpha-1} \psi\big) - \big(\nabla_k \Delta^{n-\alpha} \nabla_i \psi^*\big) \big(\Delta^{\alpha-1} \psi\big) \Big]. \tag{29}$$

Lagrangian (19) is also invariant under the phase transformation

$$\psi(x) \to \tilde{\psi}(x) = \psi(x)e^{i\alpha}, \tag{30}$$

where the parameter α does not depend on coordinates and time. A consequence of this phase symmetry is the probability conservation law

$$\left(\psi\frac{\partial L}{\partial\dot{\psi}} - \psi^*\frac{\partial L}{\partial\dot{\psi}^*}\right) + \\ +\nabla_i \sum_{n=1}^{\infty}\sum_{\alpha=1}^{n}\Big[(\nabla_i\Delta^{n-\alpha}\psi)\Delta^{\alpha-1}L_n - (\Delta^{n-\alpha}\psi)\nabla_i\Delta^{\alpha-1}L_n - \\ -(\nabla_i\Delta^{n-\alpha}\psi^*)\Delta^{\alpha-1}L_n^* + (\Delta^{n-\alpha}\psi^*)\nabla_i\Delta^{\alpha-1}L_n^*\Big] = 0. \tag{31}$$

This equation, with taking into account formulas (24), coincides with the continuity equation (7) for the probability density which is obtained directly from equation (6).

5 Interaction with the electromagnetic field

The interaction with the electromagnetic field is enabled using the well-known derivative substitution

$$\frac{\partial}{\partial x_\mu} \to \frac{\partial}{\partial x_\mu} + i\frac{e}{\hbar c}A_\mu, \tag{32}$$

where $e = \pm|e|$, $A_\mu \equiv (\mathbf{A}, A_4 = i\Phi)$ is a 4-vector potential of the electromagnetic field. In three-dimensional notation, the substitution (32) is equivalent to the following substitutions:

$$\nabla \to \nabla + \frac{e}{\hbar c}\mathbf{A}, \qquad i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} + e\Phi, \\ \Delta \to \Delta_A \equiv \Delta + \frac{e}{\hbar c}\nabla\mathbf{A} + \frac{2e}{\hbar c}\mathbf{A}\nabla + \left(\frac{e}{\hbar c}\right)^2\mathbf{A}^2. \tag{33}$$

As a result, we obtain the quantum relativistic equation for the probability amplitude in the electromagnetic field:

$$i\hbar\frac{\partial\psi}{\partial t} - U(x)\,\psi = \sqrt{m^2c^4 - \hbar^2c^2\Delta_A}\;\psi, \tag{34}$$

where $U(x) = -e\Phi(x)$ is the potential energy in an external scalar field.

6 Schrödinger equation with the relativistic correction

Let us write the Schrödinger equation with the scalar potential, taking into account the main relativistic correction. In this approximation

$$\sqrt{1-\lambda\!\!\!^{-2}\Delta}\,\psi \approx \left(1-\frac{1}{2}\lambda\!\!\!^{-2}\Delta-\frac{1}{8}\lambda\!\!\!^{-4}\Delta^2\right)\psi. \tag{35}$$

Then equation (34) will take the form

$$i\hbar\frac{\partial\psi}{\partial t}=U(x)+mc^2\left(1-\frac{1}{2}\lambda\!\!\!^{-2}\Delta-\frac{1}{8}\lambda\!\!\!^{-4}\Delta^2\right)\psi. \tag{36}$$

In order to proceed to the nonrelativistic limit, it is convenient to pass to the wave function $\chi(x)$, which differs from the original function by a time-dependent phase factor

$$\psi(x)=\chi(x)\,e^{-i\frac{mc^2}{\hbar}t}. \tag{37}$$

Then equation (36) will take the form

$$i\hbar\frac{\partial\chi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\chi+U(x)\,\chi-\frac{\hbar^4}{8m^3c^2}\Delta^2\chi. \tag{38}$$

Here the last term is the relativistic correction. The probability flux density (18) $\mathbf{j}=\mathbf{j}^{(1)}+\mathbf{j}^{(2)}$ is the sum of the nonrelativistic part $\mathbf{j}^{(1)}$ and the relativistic correction $\mathbf{j}^{(2)}$:

$$\begin{aligned}\mathbf{j}^{(1)}&=-\frac{i\hbar}{2m}(\chi^*\nabla\chi-\chi\nabla\chi^*),\\ \mathbf{j}^{(2)}&=-\frac{i\hbar}{8m}\lambda\!\!\!^{-2}(\chi^*\nabla\Delta\chi-\chi\nabla\Delta\chi^*+\Delta\chi^*\nabla\chi-\Delta\chi\nabla\chi^*).\end{aligned} \tag{39}$$

The energy density (22) and the energy flux density $\mathbf{j}_E=\mathbf{j}_E^{(1)}+\mathbf{j}_E^{(2)}$ (25) in the considered approximation are given by the formulas

$$\mathrm{H}=mc^2|\chi|^2-\frac{\hbar^2}{4m}(\chi^*\Delta\chi+\chi\Delta\chi^*)-\frac{\hbar^2}{16m}\lambda\!\!\!^{-2}(\chi^*\Delta^2\chi+\chi\Delta^2\chi^*), \tag{40}$$

$$\mathbf{j}_E^{(1)} = \frac{\hbar^2}{4m}(\dot{\chi}^*\nabla\chi + \dot{\chi}\nabla\chi^* - \chi^*\nabla\dot{\chi} - \chi\nabla\dot{\chi}^{*'} + i\frac{\hbar}{2m}mc^2(\chi^*\nabla\chi - \chi\nabla\chi^*),$$

$$\mathbf{j}_E^{(2)} = \frac{\hbar^2}{16m}\lambda\!\!\!^{-2}\Big[(\nabla\chi^*)(\Delta\dot{\chi}) + (\Delta\dot{\chi}^*)(\nabla\chi) + (\nabla\Delta\chi^*)\dot{\chi} + \dot{\chi}^*(\nabla\Delta\chi) -$$
$$-\chi^*(\nabla\Delta\dot{\chi}) - (\nabla\Delta\dot{\chi}^*)\chi - (\Delta\chi^*)(\nabla\dot{\chi}) - (\nabla\dot{\chi}^*)(\Delta\chi)\Big] -$$
$$-i\frac{\hbar^2}{8m}mc^2\lambda\!\!\!^{-2}\Big[(\nabla\chi^*)(\Delta\chi) - (\Delta\chi^*)(\nabla\chi) + (\nabla\Delta\chi^*)\chi - \chi^*(\nabla\Delta\chi)\Big]. \tag{41}$$

As we see, in the continuity equation for energy, even in the nonrelativistic limit, it is necessary to take into account the rest energy of the particle.

The momentum density (27) is expressed through the probability flux density in the nonrelativistic approximation by the relation $\boldsymbol{\pi} = m\mathbf{j}^{(1)}$. The flux density of i-component of momentum σ_{ik} is given by the formulas

$$\sigma_{ik} = \sigma_{ik}^{(1)} + \sigma_{ik}^{(2)},$$

$$\sigma_{ik}^{(1)} = \frac{\hbar^2}{4m}\Big[\chi^*(\nabla_i\nabla_k\chi) + \chi(\nabla_i\nabla_k\chi^*) - \nabla_i\chi^*\nabla_k\chi - \nabla_i\chi\nabla_k\chi^*\Big],$$

$$\sigma_{ik}^{(2)} = \frac{\hbar^2}{16m}\lambda\!\!\!^{-2} \times$$

$$\times\Big[\chi^*(\nabla_i\nabla_k\Delta\chi) + \chi(\nabla_i\nabla_k\Delta\chi^*) + \Delta\chi^*(\nabla_i\nabla_k\Delta\chi) + \Delta\chi(\nabla_i\nabla_k\Delta\chi^*) -$$
$$-(\nabla_i\Delta\chi^*)\nabla_k\chi - (\nabla_k\Delta\chi^*)\nabla_i\chi - \nabla_i\chi^*(\nabla_k\Delta\chi) - \nabla_k\chi^*(\nabla_i\Delta\chi)\Big]. \tag{42}$$

In equation (38) the last term should be considered as a small perturbation.

7 Evolution of the wave function in an unlimited space

Let us consider the evolution of the wave function in an unlimited medium. Let us move on to the new wave function $\chi(x)$ (37), such that $\psi(x) = \chi(x)\exp(-it/\tau_C)$, where $\tau_C \equiv \lambda\!\!\!^{-}/c = \hbar/mc^2$. Note that for an electron $\tau_C \approx 1.3 \cdot 10^{-21}$ s. In this case equation (6) will take the form

$$i\tau_C\dot{\chi} = \Big(\sqrt{1 - \lambda\!\!\!^{-2}\Delta} - 1\Big)\chi. \tag{43}$$

The equations for the real and imaginary parts of the wave function $\chi = (\chi' + i\chi'')/\sqrt{2}$ can be written in the form

$$\tau_C \dot{\chi}' = \left(\sqrt{1 - \lambda\!\!\!^{-2}\Delta} - 1\right)\chi'', \quad \tau_C \dot{\chi}'' = -\left(\sqrt{1 - \lambda\!\!\!^{-2}\Delta} - 1\right)\chi'. \tag{44}$$

There holds the normalization condition

$$\frac{1}{2}\int d\mathbf{x}\left[\chi'^2(\mathbf{x},t) + \chi''^2(\mathbf{x},t)\right] = 1. \tag{45}$$

Let us consider the evolution of the wave function in an unlimited space, provided that at the initial moment $t = 0$ there are given the functions $\chi'(\mathbf{x},0)$ and $\chi''(\mathbf{x},0)$, normalized by the condition (45) . From equations (44), it follows a second order equation in time for the function χ':

$$\tau_C^2 \ddot{\chi}' + \left(\sqrt{1 - \lambda\!\!\!^{-2}\Delta} - 1\right)^2 \chi' = 0. \tag{46}$$

We expand the required functions into the Fourier integral

$$\chi'(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int \chi'_k(t)\, e^{i\mathbf{k}\mathbf{x}} d\mathbf{k}, \quad \chi''(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int \chi''_k(t)\, e^{i\mathbf{k}\mathbf{x}} d\mathbf{k}. \tag{47}$$

Substituting these expansions into (46) and the second equation (44), we obtain

$$\ddot{\chi}' + \omega_k^2 \chi'_k = 0, \qquad \dot{\chi}'' + \omega_k \chi'_k = 0, \tag{48}$$

where

$$\omega_k = \left(\sqrt{1 + \lambda\!\!\!^{-2}k^2} - 1\right)\frac{1}{\tau_C}. \tag{49}$$

In the nonrelativistic limit $\lambda\!\!\!^{-}k \ll 1$

$$\omega_k \approx \frac{\lambda\!\!\!^{-2}k^2}{2\tau_C} = \frac{\hbar k^2}{2m}. \tag{50}$$

The group velocity, which determines the speed of propagation of a wave-particle

$$\nu_g = \frac{d\omega_k}{dk} = \lambda\!\!\!^{-}\frac{ck}{\sqrt{1 + \lambda\!\!\!^{-2}k^2}}, \tag{51}$$

cannot exceed the speed of light. A calculation of the group velocity within the framework of ordinary nonrelativistic quantum mechanics by formula

(50) gives $v_g = \hbar k/m$, so that with an increase of k or a decrease of the de Broglie wavelength the speed of signal propagation can be arbitrarily large.

Solutions of equations (48) have the form

$$\chi'_k(t) = A_k e^{-i\omega_k t} + B_k e^{i\omega_k t}, \qquad \chi''_k(t) = -iA_k e^{-i\omega_k t} + iB_k e^{i\omega_k t}. \tag{52}$$

From the conditions for reality of the functions χ', χ'' it follows $\chi'_k = \chi'^*_{-k}$ and $\chi''_k = \chi''^*_{-k}$, so that $B_k = A^*_{-k}$. The normalization condition (45) gives

$$\int d\mathbf{k} \left(|A_k|^2 + |A_{-k}|^2 \right) = 1. \tag{53}$$

The coefficients in (52) can be expressed through the initial conditions

$$A_k = \frac{1}{2}\left(\chi'_k(0) + i\chi''_k(0)\right), \quad A^*_{-k} = \frac{1}{2}\left(\chi'_k(0) - i\chi''_k(0)\right), \tag{54}$$

so that

$$\begin{aligned} \chi'_k(t) &= \chi'_k(0)\cos\omega_k t + \chi''_k(0)\sin\omega_k t, \\ \chi''_k(t) &= -\chi'_k(0)\sin\omega_k t + \chi''_k(0)\cos\omega_k t\,. \end{aligned} \tag{55}$$

Taking into account the relations

$$\chi'_k(0) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}} \chi'(\mathbf{x}, 0), \quad \chi''_k(0) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}} \chi''(\mathbf{x}, 0), \tag{56}$$

we find solutions which determine the real and imaginary parts of the wave function at an arbitrary moment of time from their values at the initial moment of time

$$\begin{aligned} \chi'(\mathbf{x}, t) &= \int d\mathbf{x}' \left[J_C(\mathbf{x} - \mathbf{x}', t)\chi'(\mathbf{x}', 0) + J_S(\mathbf{x} - \mathbf{x}', t)\chi''(\mathbf{x}', 0) \right], \\ \chi''(\mathbf{x}, t) &= \int d\mathbf{x}' \left[-J_S(\mathbf{x} - \mathbf{x}', t)\chi'(\mathbf{x}', 0) + J_C(\mathbf{x} - \mathbf{x}', t)\chi''(\mathbf{x}', 0) \right], \end{aligned} \tag{57}$$

where

$$\begin{aligned} J_C(\mathbf{x} - \mathbf{x}', t) &= \frac{1}{(2\pi)^3} \int d\mathbf{k} \cos\omega_k t\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}, \\ J_S(\mathbf{x} - \mathbf{x}', t) &= \frac{1}{(2\pi)^3} \int d\mathbf{k} \sin\omega_k t\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}. \end{aligned} \tag{58}$$

Using the expansion of a plane wave in terms of spherical functions and integrating over angles, we obtain

$$J_C(\mathbf{x}-\mathbf{x}',t) \equiv J_C(|\mathbf{x}-\mathbf{x}'|,t) = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 \cos(\omega_k t)\, j_0\big(k|\mathbf{x}-\mathbf{x}'|\big),$$
$$J_S(\mathbf{x}-\mathbf{x}',t) \equiv J_S(|\mathbf{x}-\mathbf{x}'|,t) = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 \sin(\omega_k t)\, j_0\big(k|\mathbf{x}-\mathbf{x}'|\big), \tag{59}$$

where $j_0(x) = \sin x/x$ is the spherical Bessel function.

Let us express the sought functions at the initial moment $t = 0$ in terms of the modulus and phase

$$\chi'(\mathbf{x},0) = \sqrt{2}\,|\psi(\mathbf{x},0)|\cos\theta(\mathbf{x},0), \quad \chi''(\mathbf{x},0) = \sqrt{2}\,|\psi(\mathbf{x},0)|\sin\theta(\mathbf{x},0). \tag{60}$$

As the initial condition we choose the Gaussian distribution for the probability density

$$|\psi(r,0)|^2 = \frac{1}{\pi^{3/2}\sigma^3}\, e^{-r^2/\sigma^2}, \tag{61}$$

where $r \equiv |\mathbf{x}|$, and the parameter σ determines the width of the wave packet, so that as σ decreases the distribution (61), satisfying the normalization condition $4\pi\int_0^\infty |\psi(r,0)|^2 r^2 dr = 1$, approaches the delta function. For the initial phase in (60) we choose $\theta(\mathbf{x},0) = 0$, so that for the real and imaginary parts we have

$$\chi'(r,0) = \sqrt{2}\,|\psi(r,0)| = \frac{\sqrt{2}}{\pi^{3/4}\sigma^{3/2}}\, e^{-r^2/(2\sigma^2)}, \quad \chi''(r,0) = 0. \tag{62}$$

In the calculation it is helpful to use the well-known integral [6]

$$\int_0^\infty e^{-a^2t^2} t^{\nu+3/2} j_{\nu-1/2}(bt)\, dt = \sqrt{\frac{\pi}{2}}\frac{b^{\nu-1/2}}{(2a^2)^{\nu+1}}\, e^{-\frac{b^2}{4a^2}}. \tag{63}$$

As a result, we obtain a solution in the form

$$\chi'(r,t) = \frac{\sqrt{2}}{\pi^{3/4}}\sqrt{\frac{2}{\pi}}\,\sigma^{3/2}\int_0^\infty dk\, k^2 \cos(\omega_k t)\, j_0(kr)\, e^{-\frac{k^2\sigma^2}{2}},$$
$$\chi''(r,t) = -\frac{\sqrt{2}}{\pi^{3/4}}\sqrt{\frac{2}{\pi}}\,\sigma^{3/2}\int_0^\infty dk\, k^2 \sin(\omega_k t)\, j_0(kr)\, e^{-\frac{k^2\sigma^2}{2}}. \tag{64}$$

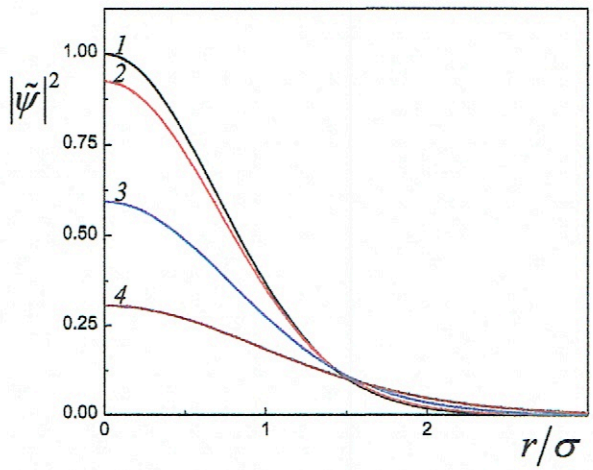

Figure 1: The spatial distribution of the probability density $|\tilde{\psi}|^2 \equiv \pi^{3/2}\sigma^3|\psi|^2$ for $\sigma/\lambda = 1$ at the moments of time $\tau = t/\tau_C$: *1* – 0; *2* – 0.5; *3* – 1.3; *4* – 2.0.

The spatial distribution of the probability density $|\tilde{\psi}|^2 \equiv \pi^{3/2}\sigma^3|\psi|^2 = \pi^{3/2}\sigma^3|\chi|^2$ at some moments of time is shown in Figure 1. As we see, a particle localized at the initial moment of time at the origin of coordinate system is gradually with equal probability spreading throughout space.

8 Relativistic correction in the oscillator theory

The Schrödinger equation for a one-dimensional along the x - axis quantum oscillator with account of the main relativistic correction (38) has the form

$$i\hbar\frac{\partial\chi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\chi}{\partial x^2} + \frac{m\omega^2x^2}{2}\chi - \frac{\hbar^4}{8m^3c^2}\frac{\partial^4\chi}{\partial x^4}. \tag{65}$$

In the stationary case, when $\chi(x,t) = \varphi(x)\exp\bigl(-iEt/\hbar\bigr)$, we have

$$E\varphi = -\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + \frac{m\omega^2x^2}{2}\varphi - \frac{\hbar^4}{8m^3c^2}\frac{d^4\varphi}{dx^4}. \tag{66}$$

It is convenient to introduce a dimensionless coordinate $\xi \equiv \left(\sqrt{m\omega/\hbar}\right)x$, and in result the equation takes the form

$$\varepsilon\varphi = -\varphi^{(II)} + \xi^2\varphi - \gamma\varphi^{(IV)}, \tag{67}$$

where differentiation occurs with respect to ξ, $\varepsilon \equiv 2E/\hbar\omega$, and

$$\gamma \equiv \frac{\hbar\omega}{4mc^2}. \tag{68}$$

In order to estimate this dimensionless parameter, we take $\hbar\omega \sim 1\,\mathrm{K} \approx 10^{-16}\,\mathrm{erg}$, $m \approx 10^{-23}\,\mathrm{g}$. As a result, we get a very small value $\gamma \sim 10^{-14}$.

The correction to n-th energy level is determined by the diagonal element of the perturbation operator taken over the unperturbed wave functions [5]

$$\varepsilon^{(1)} = -\gamma\langle n|\frac{d^4}{d\xi^4}|n\rangle. \tag{69}$$

The wave functions of an oscillator in the coordinate representation are expressed through Hermite polynomials [5]

$$\langle\xi|n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,e^{-\frac{\xi^2}{2}}H_n(\xi). \tag{70}$$

As a result, we find

$$\varepsilon^{(1)} = -\gamma\,\frac{3}{2}\,(n+1)(n+2). \tag{71}$$

Thus, the energy of the oscillator level with account of the main relativistic correction takes the form

$$E_n = \hbar\omega\left[n\left(1-\frac{9}{4}\gamma\right) + \frac{1}{2}(1-3\gamma) - \frac{3}{4}\gamma n^2\right]. \tag{72}$$

As we see, the relativistic effect leads to a decrease of the ground and excited levels, as well as to a violation of the equidistance of the spectrum. The effect of violation of the equidistance property, leading to a decrease in the distance between higher levels, can be used for the experimental detection of relativistic effects in quantum mechanics.

9 Conclusion

The paper proposes the relativistic generalization of the Schrödinger equation for the complex function, which can be interpreted as a probability amplitude. Unlike the nonrelativistic equation, this equation allows propagation of signals only at a speed not exceeding the speed of light. The theory is also formulated within the Lagrangian formalism. The equations of conservation of the probability, energy and momentum of the complex field are obtained. The problem of evolution of a wave packet in an unlimited space is solved. The relativistic corrections to the energy levels of the harmonic oscillator are found. It is shown that relativistic effects lead to a shift and a violation of the level equidistance, which can be used for experimental detection of the relativistic effect in the probabilistic quantum theory.

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